

Topological Dynamical Systems of Type I

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Abstract

The equivalence of existence of a Borel section to nonexistence of recurrent aperiodic points for homeomorphisms of locally compact topological spaces is proved using the theory of C^* -algebras.

1 Introduction

In this paper we will demonstrate that the theory of C^* -algebras can be effectively used to prove statements about topological dynamical systems.

Throughout this paper a topological dynamical system will mean, if not stated otherwise, a pair $\Sigma = (X, \sigma)$ consisting of a compact or locally compact Hausdorff topological space X and a homeomorphism $\sigma : X \rightarrow X$. The word “dynamical” is used to stress that above all we are interested in the action on X of iterations of σ . For a positive integer n , the n 'th iteration of σ is defined as a mapping $\sigma^n = \sigma \circ \cdots \circ \sigma : X \rightarrow X$ obtained by composing σ with itself n times. If $n = 0$, then $\sigma^0 = id_X : X \rightarrow X$ is the identity mapping leaving invariant all points of X . When n is a negative integer, the n 'th iteration of σ is defined as n 'th iteration of the mapping $\sigma^{-1} : X \rightarrow X$ inverse to σ . The inverse mapping σ^{-1} exists and is a homeomorphism, since σ is a homeomorphism.

The following notations will be used:

- $Per(\sigma)$ = the set of all periodic points, i.e. the points $x \in X$ such that $\sigma^n(x) = x$ for some positive integer n ;
- $Aper(\sigma)$ = $X \setminus Per(\sigma)$, the set of all aperiodic points, i.e. points which are not periodic ;
- $Per^n(\sigma)$ = the set of all periodic points with periods not greater than n , i.e. such points $x \in X$ that $\sigma^k(x) = x$ for some integer k satisfying $1 \leq k \leq n$;
- $Per_n(\sigma)$ = $Per^n(\sigma) \setminus (\bigcup_{m=1}^{n-1} Per^m(\sigma))$ for any positive integer n .

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The points in the set $Per_n(\sigma)$ are called n -periodic points or points of period n . If $x \in Per_n(\sigma)$, then n is the smallest positive integer such that $\sigma^n(x) = x$. When $n = 1$, $Per_1(\sigma) = Per^1(\sigma)$ is the set of fixed points of σ . Note also that

$$Per(\sigma) = \bigcup_{n=1}^{\infty} Per^n(\sigma) = \bigcup_{n=1}^{\infty} Per_n(\sigma),$$

where sets in the second union are pointwise disjoint. The set $Orb_\sigma(x) = \{\sigma^m(x) | m \in \mathbb{Z}\}$ is called the orbit of x . If $x \in Per_n(\sigma)$, then $Orb_\sigma(x) = \{\sigma^m(x) | 0 \leq m \leq n-1\}$ consists of n points. The set $Orb_\sigma^+(x) = \{\sigma^m(x) | m \in \mathbb{Z}, m \geq 0\}$ is called the forward orbit of x . The closure of $Orb_\sigma(x)$ and $Orb_\sigma^+(x)$ will be denoted respectively by $\overline{Orb_\sigma(x)}$ and $\overline{Orb_\sigma^+(x)}$. The mapping $k \mapsto (\sigma^k : X \rightarrow X)$ defines action of the group \mathbb{Z} on X . Two points in X are in the same coset if and only if they are in the same orbit. So the coset space X/\mathbb{Z} coincides with the orbit space of σ . The quotient mapping $x \mapsto Orb_\sigma(x)$ of X onto X/\mathbb{Z} will be denoted by Q_σ .

A point $x \in X$ will be called a *recurrent* point if there is a subsequence of the orbit of x converging to x . We will denote the set of all recurrent points by $\mathcal{C}(\sigma)$. Any periodic point is a recurrent point, because the constant sequence consisting of x converges to x and is a subsequence of the forward orbit of x .

A subset $S \subseteq X$ is called a section of a dynamical system $\Sigma = (X, \sigma)$ if S intersects any orbit at a single point. A Borel measurable section is a section which is a Borel subset of X .

Example 1. ([SW, S]) Let $\sigma(z) = \frac{a+cz}{b+dz}$ be a Möbius homeomorphism on the extended complex plane $\overline{\mathbb{C}}$ where $ad - bc \neq 0, d \neq 0$. The Möbius transformation σ has either a single fixed point or two distinct fixed points ξ_1 and ξ_2 . When σ has a single fixed point or has two distinct fixed points ξ_1 and ξ_2 , and $|q| = \left| \frac{c-\xi_1 d}{c-\xi_2 d} \right| \neq 1$, all orbits of the dynamical system generated by σ converge, and

$$Per(\sigma) = Per^1(\sigma) = \mathcal{C}(\sigma),$$

that is the set of periodic points coincides with the set of recurrent points as in this case both sets are the set of the fixed points for σ . If σ has two distinct fixed points ξ_1 and ξ_2 and $|q| = \left| \frac{c-\xi_1 d}{c-\xi_2 d} \right| = 1$, then orbits of σ different from the fixed points do not converge, but instead go around some circles. In this case the mapping σ has periodic points different from the fixed points if and only if the number $q = \frac{c-\xi_1 d}{c-\xi_2 d} \neq 1$ is an n th root of unity for some positive integer n . In this case all points different from the fixed points are periodic with the same period equal to the smallest positive integer n such that $q^n = 1$, and hence

$$Per(\sigma) = \overline{\mathbb{C}} = \mathcal{C}(\sigma).$$

Thus in this case as well the set of periodic points coincides with the set of recurrent points. Another common feature of these three classes of Möbius transformations is that each such Möbius transformation has a Borel measurable section. Typically a Borel measurable section of Möbius transformations is a region between some circles and lines united with some part of its boundary. It can be explicitly described by equations of those circles and lines directly in terms of the coefficients of the Möbius transformation. Existence of

Borel measurable sections means that the orbit space of the dynamical systems can be parameterized in a Borel measurable way, and thus is more tractable.

If a Möbius transformation belongs to neither of those three classes, that is $q = \frac{c-\xi_1 d}{c-\xi_2 d}$ is on the unit circle but is not a root of unity, then the only periodic points are the fixed points. The orbits of other points are dense subsets of some circles. In this case, there is no Borel measurable section and at the same time

$$Per(\sigma) = Per^1(\sigma) \neq \mathcal{C}(\sigma) = \overline{\mathcal{C}}.$$

A conclusion of this example is that for dynamical systems generated by such important class of mappings as Möbius transformations the existence of a Borel measurable section is equivalent to the equality $\mathcal{C}(\sigma) = Per(\sigma)$.

Example 2. Let $\sigma(z) = e^{i2\pi\theta}z$ be a rotation of the one-dimensional torus (the unite circle) \mathbb{T} by some angle $2\pi\theta$, where $\theta \in \mathbb{R}$. If θ is rational, that is $\theta = \frac{k}{n}$ with k and n being coprime integers and $n > 0$, then every point of \mathbb{T} is periodic for σ with period n , and thus is recurrent. When θ is irrational, the orbit of any point of \mathbb{T} is dense in \mathbb{T} , and thus any point is recurrent but not periodic. So, $Per(\sigma) = \mathcal{C}(\sigma) = \mathbb{T}$ for rational θ , and $Per(\sigma) = \emptyset \neq \mathcal{C}(\sigma) = \mathbb{T}$ for irrational θ . Moreover, for rational $\theta = \frac{k}{n}$ with k and n being coprime integers and $n > 0$, each orbit consists of n points coinciding with n 'th roots of 1, and the interval between two neighbor n 'th roots of 1 with one of the ends excluded is a Borel measurable section for the corresponding dynamical system as it is evidently a Borel set having exactly one point in common with each orbit. On the other hand, when θ is irrational, a set containing a single point from each orbit gives the standard example of not measurable set, which means that for irrational θ no Borel measurable section exists. So, for rotations of the circle, as for the Möbius transformations, the property of all recurrent points being periodic, that is $Per(\sigma) = \mathcal{C}(\sigma)$, is equivalent to existence of a Borel measurable section.

Example 3. Let $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ be a homeomorphism of the circle. Then either σ is orientation preserving or orientation reversing. If σ is orientation reversing, then it has a fixed point. If σ is orientation preserving, then it might be with or without periodic points. Thus all homeomorphisms of the circle are subdivided in three non-overlapping classes – orientation preserving homeomorphisms without periodic points ($Per(\sigma) = \emptyset$), orientation preserving homeomorphisms with non-empty set of periodic points ($Per(\sigma) \neq \emptyset$), and orientation reversing homeomorphisms with non-empty set of periodic points.

As \mathbb{T} is compact, the set of recurrent points $\mathcal{C}(\sigma)$ is non-empty, and its closure coincides with the Birkhoff center of the dynamical system (X, σ) , for any homeomorphism σ of \mathbb{T} .

If σ is a homeomorphism of the circle, then it is topologically conjugate to the irrational rotation if and only if it is orientation preserving, and it has no periodic points, but there is a point with dense orbit in \mathbb{T} . So, for such homeomorphisms, there is no Borel measurable section, and at the same time not all recurrent points are periodic as the set of periodic points is empty while the set of recurrent points contains the point with dense orbit. Thus for order preserving homeomorphisms of \mathbb{T} which have no periodic points but possess a point with a dense orbit, the property of all recurrent points being periodic, is equivalent to existence of a Borel measurable section (see also Remark 3).

A natural question arising from examples 1, 2 and 3 is whether existence of Borel measurable section is equivalent to $\mathcal{C}(\sigma) = \text{Per}(\sigma)$ for more general classes of dynamical systems.

The main result of this article on the side of dynamical systems is the following general theorem.

Theorem 1. *Let X be either a compact Hausdorff topological space satisfying the second axiom of countability, or locally compact Hausdorff topological space whose Alexandroff's one-point compactification satisfies the second axiom of countability, and let $\sigma : X \rightarrow X$ be a homeomorphism of X . Then the following two properties of the dynamical system $\Sigma = (X, \sigma)$ are equivalent:*

- 1) *there exists a Borel section for Σ ;*
- 2) $\mathcal{C}(\sigma) = \text{Per}(\sigma)$.

As surprising this might seem, this useful result in such generality seems to be unavailable in the literature on topological dynamical systems.

The main purpose of this article is to show how the theory of C^* -algebras can be effectively used to prove general results on dynamical systems such as Theorem 1.

Remark 1. Any locally compact Hausdorff topological space X satisfying the second axiom of countability is metrizable. Whenever the second axiom of countability is satisfied by the Alexandroff's one-point compactification of a locally compact Hausdorff topological space X , then it is also satisfied by the space X itself thus making it metrizable.

Remark 2. It is sufficient to prove Theorem 1 when X is a compact Hausdorff topological space satisfying the second axiom of countability. Indeed, if X is locally compact Hausdorff topological space whose Alexandroff's one-point compactification $\tilde{X} = X \cup \{\omega\}$ satisfies the second axiom of countability, and σ is a homeomorphism of X , then the extension $\tilde{\sigma}$ of σ onto \tilde{X} leaving fixed the compactifying point $\{\omega\}$ is a homeomorphism of \tilde{X} such that $\text{Per}(\tilde{\sigma}) = \text{Per}(\sigma) \cup \{\omega\}$ and $\mathcal{C}(\tilde{\sigma}) = \mathcal{C}(\sigma) \cup \{\omega\}$. The existence of Borel measurable section takes place simultaneously for $\tilde{\sigma}$ and σ , as M is a Borel measurable section for σ if and only if $M \cup \{\omega\}$ is a Borel measurable section for $\tilde{\sigma}$.

Remark 3. By theorem 1, if a homeomorphism σ of a compact Hausdorff space X satisfying the second axiom of countability has no periodic points, then the dynamical system $\Sigma = (X, \sigma)$ has no Borel measurable section, since any homeomorphism of a compact space has a recurrent point. In particular, the homeomorphisms of the circle \mathbb{T} which have no periodic points have no Borel measurable section, independently on whether they possess a point with a dense orbit or not.

Remark 4. Theorem 1 implies moreover, that the dynamical system generated by any topologically transitive homeomorphism (with or without periodic points) of infinite compact Hausdorff space satisfying the second axiom of countability has no Borel measurable section. Indeed, any such homeomorphism has a point with dense orbit which can not be periodic, because being dense in the infinite topological space with those properties, it has to be infinite.

Remark 5. If some restriction of a given dynamical system onto a Borel measurable subset satisfies conditions of the theorem, and has recurrent points which are not periodic, then the whole system has no Borel measurable section. Such situation occurs for instance in the dynamical systems generated by Möbius transformations of non-periodic divergence type mentioned in Example 1.

More detailed discussion of the dynamics of Möbius transformations in line with Example 2, as well as applications to representations by operators on Hilbert spaces of the corresponding class of crossed product algebras, can be found in [SW] and [S]. Many results and further references on the dynamics of homeomorphisms of the circle without periodic points are contained in the works by A. Denjoy [De], H. Furstenberg [F], N. G. Markley [Ma1, Ma2, Ma3], as well as in the books by Ya. G. Sinai [Sin] and J. de Vries [Vr]. The homeomorphisms of the circle with non-empty set of periodic points are considered in the work of A. DeRango [DR], devoted to classification and representations of the corresponding crossed product C^* -algebras. The book of J. de Vries [Vr] contains a lot of material and references devoted to recurrence and periodicity in dynamical systems, providing rich variety of concrete examples of classes of dynamical systems to which Theorem 1 can be successfully applied.

2 Topological dynamics and C^* -algebras

In section 3 we will prove Theorem 1 using the theory of C^* -algebras and their representations. We attempted to make the proof as accessible as possible for specialists in dynamical systems. Thus we firstly discuss some facts and definitions on C^* -algebras necessary for understanding the proof. We recommend the books [BR], [Dav], [Dix2], [KR], [Li], [Mur], [Ped], [Sak], [Tak], [Tom1] for detailed expositions of the theory of C^* -algebras.

Let $\Sigma = (X, \sigma)$ be a topological dynamical system which satisfies the conditions in Theorem 1.

The $*$ -algebra of all continuous functions on X and the $*$ -algebra of all continuous functions on X with compact support will be denoted respectively by $C(X)$ and by $C_c(X)$. The algebra $C(X)$ has a unit if and only if X is compact, and the unit then is the constant function $\mathbf{1} = \mathbf{1}_{C(X)}(\cdot)$ equal to 1 on all elements of X . Moreover, X is compact if and only if $C(X)$ and $C_c(X)$ coincide.

The mapping $\alpha : C(X) \rightarrow C(X)$ defined by

$$\alpha(f)(x) = f(\sigma^{-1}(x)) \quad (1)$$

is an automorphism of the $*$ -algebra $C(X)$, and the mapping defined by

$$j \mapsto \alpha^j(f)(x) = f(\sigma^{-j}(x)) \quad (2)$$

is a homomorphism of \mathbb{Z} into the group $\text{Aut}(C(X))$ of automorphisms of $C(X)$. Since σ is a homeomorphism, the family of all compact subsets of X is invariant with respect to σ and σ^{-1} , and hence α leaves invariant the $*$ -subalgebra $C_c(X)$ of $C(X)$. The group \mathbb{Z} is a locally compact group with respect to the discrete topology, i.e. the topology where any

subset of \mathbb{Z} is open. A subset of \mathbb{Z} is compact if and only if it is finite. The set $C_c(\mathbb{Z}, C(X))$ of continuous mappings from \mathbb{Z} to $C(X)$ with compact support consists of all mappings which may assume non-zero values only at finitely many elements of \mathbb{Z} . For any function $a : \mathbb{Z} \rightarrow C(X)$ we denote by $a[k]$ the element of $C(X)$ equal to the value of a at $k \in \mathbb{Z}$. The pointwise addition and multiplication by complex numbers makes $C_c(\mathbb{Z}, C(X))$ into a linear space, which becomes a normed $*$ -algebra with the multiplication, involution and norm defined by

$$(ab)[k](\cdot) = \sum_{s \in \mathbb{Z}} a[s](\cdot) \alpha^s(b[-s+k])(\cdot) = \quad (3)$$

$$= \sum_{s \in \mathbb{Z}} a[s](\cdot) b[-s+k](\sigma^{-s}(\cdot))$$

$$b^*[k](\cdot) = \alpha^k(\overline{b[-k]})(\cdot) = \overline{b[-k]}(\sigma^{-k}(\cdot)) \quad (4)$$

$$\|b\| = \sum_{s \in \mathbb{Z}} \|b[s]\|_{C(X)} \quad (5)$$

The Banach $*$ -algebra obtained as the completion of this normed $*$ -algebra is denoted by $l^1(\mathbb{Z}, C(X))$.

Let us first consider the case when X is compact. Then $C_c(X)$ coincides with $C(X)$. The $*$ -algebra $C(X)$ becomes a C^* -algebra with respect to the supremum norm defined by $\|f\| = \|f\|_{C(X)} = \sup\{f(x) \mid x \in X\}$ for all $f \in C(X)$. The mappings defined by

$$\delta_j[k](\cdot) = \begin{cases} \mathbf{1} = \mathbf{1}_{C(X)}(\cdot) & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

for $j \in \mathbb{Z}$ belong to $C_c(\mathbb{Z}, C(X))$, and δ_0 is the unit of $C_c(\mathbb{Z}, C(X))$ and hence of $l^1(\mathbb{Z}, C(X))$. With the multiplication defined by (3), the equality $\delta_j = \delta_1^j$ holds for all $j \in \mathbb{Z} \setminus \{0\}$. In what follows, for the brevity of notations, we will denote δ_1 by δ , will assume that $\delta^0 = \delta_0$, and will write δ^j instead of δ_j for all $j \in \mathbb{Z}$. The algebra $C_c(\mathbb{Z}, C(X))$ then coincides with the algebra of polynomials in δ with coefficients in $C(X)$.

The C^* -algebra $C(X)$ is isomorphic to the C^* -subalgebra $C(X)\delta^0$ of $C_c(\mathbb{Z}, C(X))$ and of $l^1(\mathbb{Z}, C(X))$ having the same unit δ^0 . The mapping $i_0 : C(X) \rightarrow C(X)\delta^0$ sending $f \in C(X)$ to $f\delta^0 \in C_c(\mathbb{Z}, C(X))$ is a unital $*$ -isomorphism of C^* -algebra $C(X)$ onto C^* -algebra $C(X)\delta^0$. We use notation

$$(f\delta^0)[k](x) = (\delta^0 f)[k](x) = \begin{cases} f(x), & k = 0 \\ 0, & k \neq 0 \end{cases}.$$

In general, whenever it is convenient, for $a \in l^1(\mathbb{Z}, C(X))$ and $f \in C(X)$, by equalities of the form $a = f$ we will mean $a = f\delta^0$, and the notations $af = a(f\delta^0)$ and $fa = (f\delta^0)a$ will be used with products between a and $f\delta^0$ defined by (3). The same notations will be often used for a belonging to the C^* -crossed product algebra of $C(X)$ by \mathbb{Z} obtained as completion of $l^1(\mathbb{Z}, C(X))$ with respect to a certain norm. With those notations, the fundamental equality

$$\delta f \delta^* = \alpha(f), \quad (6)$$

called *covariance relation*, holds for all $f \in C(X)$.

The mapping $E : l^1(\mathbb{Z}, C(X)) \rightarrow C(X)\delta^0$ defined by $E(b) = b[0]\delta^0$ for any element $b \in l^1(\mathbb{Z}, C(X))$ is a projection of norm one satisfying

$$E(abc) = aE(b)c \text{ for all } a, c \in C(X)\delta^0, \quad (\text{module property}) \quad (7)$$

$$E(b^*b) \geq 0, \quad (\text{positivity}) \quad (8)$$

$$E(b^*b) = 0 \text{ implies that } b = 0. \quad (\text{faithfulness}) \quad (9)$$

for all $b \in l^1(\mathbb{Z}, C(X))$. The positivity, for example, is proved as follows:

$$\begin{aligned} E(b^*b) &= (b^*b)[0]\delta^0 = \left(\sum_{k \in \mathbb{Z}} b^*[k](\cdot)\alpha^k(b[-k])(\cdot)\right)\delta^0 \\ &= \left(\sum_{k \in \mathbb{Z}} \alpha^k(\overline{b[-k]}b[-k])(\cdot)\right)\delta^0 = \left(\sum_{k \in \mathbb{Z}} \alpha^k(|b[-k]|^2(\cdot))\right)\delta^0 \\ &= \sum_{k \in \mathbb{Z}} (|b[-k](\sigma^{-k}(\cdot))|^2)\delta^0 \geq 0 \end{aligned}$$

where the sums converge in norm.

For any linear functional φ on $C(X)$, the mapping $\varphi \circ i_0^{-1}$ is a linear functional on $C(X)\delta^0$ satisfying $(\varphi \circ i_0^{-1})(i_0(a)) = \varphi(a)$ for any $a \in C(X)$. Since the mapping $a \mapsto i_0(a)$ is an isometric $*$ -isomorphism of $C(X)$ onto $C(X)\delta^0$, it follows that $\|\varphi \circ i_0^{-1}\| = \|\varphi\|$ for any bounded φ on $C(X)$, and that φ is positive on $C(X)$ if and only if $\varphi \circ i_0^{-1}$ is positive on $C(X)\delta^0$.

For any positive linear functional φ on $C(X)$, the mapping $(\varphi \circ i_0^{-1}) \circ E$ is a positive linear functional on $l^1(\mathbb{Z}, C(X))$. Moreover, $\|\varphi\| = \varphi(e)$ for any positive linear functional φ on a Banach $*$ -algebra with the unit e . Since

$$\begin{aligned} \|(\varphi \circ i_0^{-1}) \circ E\| &= (\varphi \circ i_0^{-1})(E(\delta^0)) = (\varphi \circ i_0^{-1})(\delta^0) = \\ &= (\varphi \circ i_0^{-1})(i_0(\mathbf{1}_{C(X)})) = \varphi(\mathbf{1}_{C(X)}) = \|\varphi\|, \end{aligned}$$

the functional φ is a state on $C(X)$, i.e. a positive linear functional with $\|\varphi\| = 1$, if and only if $(\varphi \circ i_0^{-1}) \circ E$ is a state on $l^1(\mathbb{Z}, C(X))$.

A set of states on a Banach $*$ -algebra \mathcal{A} is said to contain sufficiently many states if for any non-zero $a \in \mathcal{A}$ there exists a state φ from this set such that $\varphi(a^*a) \neq 0$.

There are sufficiently many states on any C^* -algebra, and in particular on $C(X)$ and on its isomorphic copy $C(X)\delta^0$. By faithfulness of the projection E , the set

$$\{(\varphi \circ i_0^{-1}) \circ E \mid \varphi \text{ is a state on } C(X)\}$$

of states on $l^1(\mathbb{Z}, C(X))$ contains sufficiently many states. As a Banach $*$ -algebra with sufficiently many states, $l^1(\mathbb{Z}, C(X))$ has sufficiently many representations, i.e. for any non-zero $b \in l^1(\mathbb{Z}, C(X))$ there is a representation π with $\pi(b) \neq 0$. Then it can be shown that

$$\|b\|_\infty = \sup\{\|\pi(b)\| \mid \pi \text{ is a representation of } l^1(\mathbb{Z}, C(X))\} \leq \|b\|_1$$

defines a C^* -norm on $l^1(\mathbb{Z}, C(X))$.

The C^* -algebra obtained by completion of $l^1(\mathbb{Z}, C(X))$ with respect to the norm $\|\cdot\|_\infty$ is called the C^* -crossed product of $C(X)$ by \mathbb{Z} with respect to the action of α , or the

transformation group C^* -algebra associated with the dynamical system $\Sigma = (X, \sigma)$. Depending on which of those two terminologies is used this algebra is denoted either by $C(x) \rtimes_{\alpha} \mathbb{Z}$ or by $\mathcal{A}(\Sigma)$. The C^* -algebra $\mathcal{A}(\Sigma)$ coincides with the closed linear span of all polynomial expressions built of δ , $\delta^* = \delta^{-1}$ and also of elements from $C(X)$, or to be more precise from $C(X)\delta^0$. Because of the covariance relation (6), all δ and $\delta^* = \delta^{-1}$ in any such polynomial expression can be moved to the right of all elements of $C(X)$. Thus any polynomial expression built of δ , $\delta^* = \delta^{-1}$ and of elements of $C(X)$ is equal to a generalized polynomial in δ , that is to an element of the form $\sum_{j=-n}^{j=n} f_j \delta^j$. Consequently, the C^* -algebra $\mathcal{A}(\Sigma)$ can be viewed as a closed linear span of generalized polynomials in δ over $C(X)$. The projection E can be extended from $l^1(\mathbb{Z}, C(X))$ to $\mathcal{A}(\Sigma) = C(X) \rtimes_{\alpha} \mathbb{Z}$ with the property of being faithful and with $\|E\| = 1$. For an element a of $\mathcal{A}(\Sigma)$, the n 'th generalized Fourier coefficient $a(n)$ is defined as $E(a(\delta^*)^n)$.

If π is a representation of the C^* -algebra $\mathcal{A}(\Sigma)$ on a Hilbert space H_{π} , then $\pi' = \pi \circ i_0$ is a representation of $C(X)$ on H_{π} , and if π' is a representation of the C^* -algebra of $C(X)$ on H_{π} , then $\pi = \pi' \circ i_0^{-1}$ is a representation of $C(X)\delta^0$ on H_{π} . Moreover, $\pi'(f) = (\pi \circ i_0)(f) = \pi(f\delta^0)$ for any $f \in C(X)$. With this in mind, for simplicity of notations, if π is a representation of $\mathcal{A}(\Sigma)$ and $f \in C(X)$, then by $\pi(f)$ we will always mean $\pi(f \circ \delta^0)$.

If π is a representation of the C^* -algebra $\mathcal{A}(\Sigma)$ on a Hilbert space H_{π} , then the unitary operator $u = \pi(\delta)$ and the commutative set (algebra) of bounded operators $\pi(C(X))$ on H_{π} satisfy the set of commutation relations

$$u\pi(f)u^* = \pi(\alpha(f)) \quad (10)$$

called covariance relations or covariance relations for a set of operators, as they are obtained by applying the representation π to both sides of the covariance relation (6) in the algebra $\mathcal{A}(\Sigma)$. A pair (π, u) consisting of a representation of the C^* -algebra $C(X)$ on a Hilbert space H , and a unitary operator u on H satisfying the covariance relations (10) is called a covariant representation of the system $(C(X), \alpha, \mathbb{Z})$. So, any representation of the C^* -algebra $\mathcal{A}(\Sigma) = C(X) \rtimes_{\alpha} \mathbb{Z}$ gives rise, via restriction, to a covariant representation of the system $(C(X), \alpha, \mathbb{Z})$. Moreover, this covariant representation of $(C(X), \alpha, \mathbb{Z})$ defines uniquely the representation of $\mathcal{A}(\Sigma)$, and every covariant representation of the system $(C(X), \alpha, \mathbb{Z})$ is obtained by restriction from a representation of the C^* -algebra $\mathcal{A}(\Sigma) = C(X) \rtimes_{\alpha} \mathbb{Z}$. In other words, there is one-to-one correspondence between covariant representations of the system $(C(X), \alpha, \mathbb{Z})$, and representations of the C^* -algebra $\mathcal{A}(\Sigma) = C(X) \rtimes_{\alpha} \mathbb{Z}$. Thus the representations of the C^* -crossed product $\mathcal{A}(\Sigma) = C(X) \rtimes_{\alpha} \mathbb{Z}$ can be completely described and studied in terms of the covariant representation of the system $(C(X), \alpha, \mathbb{Z})$, that is in terms of families of operators satisfying the covariance commutation relations (10). If (π, u) is a covariant representation of the system $(C(X), \alpha, \mathbb{Z})$, then the corresponding representation of the crossed product C^* -algebra $C(X) \rtimes_{\alpha} \mathbb{Z}$ transforms a generalized polynomial $\sum_{j=-n}^n f_j \delta^j$ into the operator $\sum_{j=-n}^n \pi(f_j) u^j$.

The theory of C^* -algebra $\mathcal{A}(\Sigma)$ can be successfully used to investigate the topological dynamical system $\Sigma = (X, \sigma)$. We will show how it can be applied to prove Theorem 1.

A way to relate the dynamical system Σ to the structure of the corresponding C^* -algebra $\mathcal{A}(\Sigma)$ is to consider extensions of states on $C(X)\delta^0$ to states on $\mathcal{A}(\Sigma)$ together with GNS representations of $\mathcal{A}(\Sigma)$.

For a C^* -algebra \mathcal{A} , the set $B(0, 1; \mathcal{A}_+^*)$ of positive linear functionals on \mathcal{A} with norm not greater than 1 is a convex subset of the unit ball of the dual space \mathcal{A}^* compact with respect to the weak* topology on \mathcal{A}^* , that is a topology with a basis of neighborhoods of a point $\varphi_0 \in \mathcal{A}^*$ consisting of the sets of the form

$$W(\varphi_0; \varepsilon) = \{\varphi \in \mathcal{A}^* \mid |\varphi(a_j) - \varphi_0(a_j)| < \varepsilon \text{ for all } j = 1, 2, \dots, n\}$$

with $\varepsilon > 0$ and $\{a_1, a_2, \dots, a_n\} \subset \mathcal{A}$ being a finite subset of \mathcal{A} , or equivalently the weakest topology with respect to which the mapping $\varphi \mapsto \varphi(a)$ of \mathcal{A}^* to \mathbb{C} is continuous for any $a \in \mathcal{A}$. The set $S(\mathcal{A})$ of all states on \mathcal{A} is convex, and it is weak* compact if and only if \mathcal{A} is unital, which is the case for $C(X)$ and $\mathcal{A}(\sum)$ when X is compact. A positive linear functional φ on a C^* -algebra \mathcal{A} is called pure if any positive linear functional ψ for which $\varphi - \psi$ is positive has the form $\lambda\varphi$ with $0 \leq \lambda \leq 1$. The set of all extreme points of $B(0, 1; \mathcal{A}_+^*)$ is the union of the set $P(\mathcal{A})$ of all pure states and the zero functional. If \mathcal{A} is a unital C^* -algebra, then $P(\mathcal{A})$ coincides with the set of extreme points of the set $S(\mathcal{A})$ of all states, and $S(\mathcal{A})$ coincides with the weak* closure of the convex hull of $P(\mathcal{A})$.

For $x \in X$, the pure state on $C(X)$ defined by $\mu_x(f) = f(x)$ is called a point evaluation, or a point evaluation on $C(X)$. The space X is embedded into $S(C(X))$ as the compact subset consisting of point evaluations μ_x for $x \in X$. The pure state $\mu_x \circ i_0^{-1}$ on $C(X)\delta^0$ is also called the point evaluation, or the point evaluation on $C(X)\delta^0$.

Any positive linear functional on a C^* -subalgebra of a C^* -algebra can be extended to a positive linear functional on the whole C^* -algebra in a norm preserving way. In particular a state on a C^* -subalgebra can be extended to a state on the whole C^* -algebra. Moreover, a pure state on a C^* -subalgebra can be extended to a pure state on the whole C^* -algebra.

There is the unique pure state extension of a point evaluation on $C(X)\delta^0$ to a pure state on $\mathcal{A}(\sum)$ if and only if x is an aperiodic point. When $x \in X$ is a periodic point for σ , the pure state extensions of $\mu_x \circ i_0^{-1}$ are parameterized by the torus. The proofs of the last two facts can be found in [Tom1].

If the pure state extension of $\mu_x \circ i_0^{-1}$ is unique, then its state extension is also unique, since the set of the state extensions is the weak* closed convex hull of its extreme points, pure state extensions. If the pure state extension of $\mu_x \circ i_0^{-1}$ is unique, then it must be $\varphi_x = (\mu_x \circ i_0^{-1}) \circ E$. This means that, in particular, $\varphi_x(f\delta^n) = 0$ for any non-zero $n \in \mathbb{Z}$, and

$$\varphi_x(f\delta^0) = \varphi_x(f\delta^0) = (\mu_x \circ i_0^{-1})(E(f\delta^0)) = (\mu_x \circ i_0^{-1})(f\delta^0) = \mu_x(f) = f(x).$$

For any positive linear functional φ on a C^* -algebra \mathcal{A} , there exists a Hilbert space H_φ , a representation π_φ of \mathcal{A} on H_φ and an element $\xi_\varphi \in H_\varphi$ such that $\pi_\varphi(\mathcal{A})\xi_\varphi$ is dense in H_φ , the equality $\varphi(a) = (\pi_\varphi(a)\xi_\varphi, \xi_\varphi)$ holds for any $a \in \mathcal{A}$, and $\|\xi_\varphi\|^2 = \|\varphi\|$. Moreover, $(\pi_\varphi, H_\varphi, \xi_\varphi)$ is defined by the first two properties uniquely up to unitary equivalence, which means that for any $(\pi'_\varphi, H'_\varphi, \xi'_\varphi)$ with the same properties there exists an isometric linear mapping U of H_φ onto H'_φ such that $U\xi_\varphi = \xi'_\varphi$ and $U\pi_\varphi(a) = \pi'_\varphi(a)U$. A triple $(\pi_\varphi, H_\varphi, \xi_\varphi)$ with those properties can be constructed using the GNS (Gelfand-Naimark-Segal) construction. This triple is called the GNS representation of \mathcal{A} .

A representation of a C^* -algebra \mathcal{A} on a Hilbert space H is said to be irreducible if there is no proper $\pi(\mathcal{A})$ -invariant closed subspace of H . If π is a representation of a

C^* -algebra \mathcal{A} , then $[\pi]$ will denote the class of all representations of \mathcal{A} which are unitarily equivalent to π , and $\tilde{q} : \text{Rep}(\mathcal{A}) \rightarrow \text{Rep}(\mathcal{A})/\sim$ will denote the quotient mapping $\pi \mapsto [\pi]$ of the set $\text{Rep}(\mathcal{A})$ of all representations of \mathcal{A} onto the set $\text{Rep}(\mathcal{A})/\sim$ of equivalence classes of $\text{Rep}(\mathcal{A})$ under the unitary equivalence. The restriction of \tilde{q} onto $\text{Irr}(\mathcal{A})$ will be denoted by q . The set $\hat{\mathcal{A}} = q(\text{Irr} \mathcal{A} \setminus \{0_{\text{Irr}(\mathcal{A})}\})$ consists of all unitary equivalence classes of non-zero irreducible representations of \mathcal{A} and is often called spectrum of \mathcal{A} . Here, $0_{\text{Irr}(\mathcal{A})}$ denotes the zero irreducible representation, which obviously is one-dimensional by irreducibility. The restriction of q onto the set $\text{Irr}_n(\mathcal{A}) \setminus \{0_{\text{Irr}(\mathcal{A})}\}$ of non-zero n -dimensional irreducible representations will be denoted by q_n . The sets

$$\begin{aligned}\hat{\mathcal{A}}_n &= q_n(\text{Irr}_n(\mathcal{A}) \setminus \{0_{\text{Irr}(\mathcal{A})}\}), \\ \hat{\mathcal{A}}_{<\infty} &= \bigcup_{1 \leq k < \infty} \hat{\mathcal{A}}_k, \\ \hat{\mathcal{A}}_{\leq n} &= \bigcup_{1 \leq k \leq n} \hat{\mathcal{A}}_k \\ \hat{\mathcal{A}}_\infty &= \hat{\mathcal{A}} \setminus \hat{\mathcal{A}}_{<\infty} = q(\text{Irr}(\mathcal{A})_\infty)\end{aligned}$$

consist of all unitary equivalence classes of non-zero irreducible representations of dimension n , of a finite dimension, of a dimension not greater than n and of unitary equivalence classes of infinite-dimensional representations of a C^* -algebra \mathcal{A} respectively.

A C^* -algebra \mathcal{A} is said to be of type I if for every non-zero irreducible representation (π, H) , the image $\pi(\mathcal{A})$ contains the algebra of all compact operators $K(H)$. The C^* -algebras of type I are also called GCR- C^* -algebras, postliminal C^* -algebras or smooth C^* -algebras. If \mathcal{A} is a C^* -algebra of type I, then irreducible representations of \mathcal{A} with the same kernels are unitarily equivalent. A proof of this theorem can be found for example in [Sak], Proposition 4.6.17, or in [Dix2], Theorem 4.3.7. When a C^* -algebra is separable it was proved in [Dix1] and [Gl1] that the converse is also true, that is if every two irreducible representations with the same kernel are unitarily equivalent, then the C^* -algebra is of type I. A proof of this statement can be also found in [Dix2], Theorem 9.1. It is unknown whether, this is true for non-separable C^* -algebras. Some results and references in this direction can be also found in the book [Ped] – a classical source on crossed product C^* -algebras, and on general theory of C^* -algebras and their representations.

The GNS representation $(\pi_\varphi, H_\varphi, \xi_\varphi)$ is irreducible if and only if φ is pure. The functional φ is a state if and only if $\|\xi_\varphi\| = 1$. The mapping $\gamma : P(\mathcal{A}) \rightarrow \hat{\mathcal{A}}$, sending any pure state φ to the unitary equivalence class $[\pi_\varphi]$ of the GNS representation π_φ , is called the GNS mapping. The GNS mapping is surjective, that is for any non-zero irreducible representation (π, H) of a C^* -algebra \mathcal{A} there exists a pure state φ such that π is unitarily equivalent to the GNS representation associated with φ . The proof of this fact can be found for example in [Dix2].

Since any irreducible representation of the commutative C^* -algebra $C(X)$ is one-dimensional, the set of all pure states on $C(X)$ coincides with the set of all characters on $C(X)$, that is with the set of all non-zero linear multiplicative functionals on $C(X)$. The same is also true for the C^* -algebra $C(X)\delta^0$ isomorphic to $C(X)$, or in general for any commutative C^* -algebra.

For a point $x \in X$ and a state extension $\varphi \in S(\mathcal{A}(\Sigma))$ of the point evaluation $\mu_x \circ i_0^{-1} \in S(C(X)\delta^0)$, let $(\pi_\varphi, H_\varphi, \xi_\varphi)$ be the GNS representation of $\mathcal{A}(\Sigma)$ associated with φ . The following statements, collected here into a single theorem for convenience, describe the pure state extensions of point evaluations and the corresponding GNS representations. Their proofs can be found in [Tom2].

Theorem 2. *Let $\varphi \in S(\mathcal{A}(\Sigma))$ be a state extension of the point evaluation $\mu_x \circ i_0^{-1} \in S(C(X)\delta^0)$, and let $(\pi_\varphi, H_\varphi, \xi_\varphi)$ be the GNS representation of $\mathcal{A}(\Sigma)$ associated with φ .*

1. *Let $u = \pi_\varphi(\delta)$. Then for any $f \in C(X)$ the operators $\pi_\varphi(f)$ and u satisfy the covariance relation*

$$u\pi_\varphi(f)u^* = \pi_\varphi(f \circ \sigma^{-1})$$

2. *Define the subspace*

$$H_n = \{\xi \in H_\varphi \mid \pi_\varphi(f)\xi = f(\sigma^n x)\xi \text{ for every } f \in C(X)\}.$$

Then $\xi_\varphi \in H_0$.

If x is aperiodic then the subspaces H_n and H_m are orthogonal when $n \neq m$. If x is p -periodic, then H_n and H_m are orthogonal when $n - m$ is not divisible by p .

3. *If x is aperiodic, then*

$$H_\varphi = \sum_{n \in \mathbb{Z}} \oplus H_n \quad (\text{orthogonal sum}),$$

and $H_n = u^n H_0$ for all $n \in \mathbb{Z}$, that is u acts as a shift along the subspaces of this decomposition.

If x is p -periodic, then

$$H_\varphi = H_0 \oplus H_1 \oplus \cdots \oplus H_{p-1} \quad (\text{orthogonal sum}),$$

where $H_n = u^n H_0$ for all $n \in \{0, \dots, p-1\}$, and $H_0 = uH_{p-1} = u^p H_0$, or in other words u acts as a cyclic shift along the subspaces of this decomposition.

4. *The GNS representation π_φ is irreducible, or equivalently the state φ is pure, if and only if H_0 is one-dimensional, that is $H_0 = \{\lambda \xi_\varphi \mid \lambda \in \mathbb{C}\}$.*
5. *If the GNS representation π_φ is irreducible, and x is a periodic point of period p , then $\dim(H_\varphi) = p$, the equality $u^p \xi_\varphi = t \xi_\varphi$ holds for some $t \in \mathbb{C}$ with $|t| = 1$, and also $\varphi(\delta^{pk}) = t^k$ for all $k \in \mathbb{Z}$ and $\varphi(\delta^l) = 0$ for all $l \in \mathbb{Z}$ not divisible by p . So the mapping $(x, t) \mapsto \varphi$ is a surjection from $\text{Per}(\sigma) \times \mathbb{T}$ onto the set of pure state extensions of $\mu_x \circ i_0^{-1}$ for $x \in \text{Per}(\sigma)$. Moreover, for each $x \in \text{Per}(\sigma)$ the mapping sending every $t \in \mathbb{T}$ to the pure state φ defined by the pair (x, t) is a bijection of \mathbb{T} onto the set of pure state extensions of $\mu_x \circ i_0^{-1}$. So, the pure state extensions of the point evaluation at every periodic point are parameterized by the torus \mathbb{T} .*

If x is an aperiodic point, then the pure state extension of $\mu_x \circ i_0^{-1}$ is uniquely determined by x and thus must be $(\mu_x \circ i_0^{-1}) \circ E$.

6. If $\varphi = \varphi_{x,t}$ and $\psi = \psi_{y,s}$ are the pure state extensions of the point evaluations at periodic points x and y corresponding to $t, s \in \mathbb{T}$, then the GNS representations π_φ and π_ψ are unitarily equivalent if and only if $\text{Orb}_\sigma(x) = \text{Orb}_\sigma(y)$ and $s = t$. If $\varphi = \varphi_x$ and $\psi = \psi_y$ are pure state extensions of the point evaluations at aperiodic points x and y , then the GNS representations π_φ and π_ψ are unitarily equivalent if and only if $\text{Orb}_\sigma(x) = \text{Orb}_\sigma(y)$.

The pure state extension of the point evaluation $\mu_x \circ i_0^{-1}$ and the corresponding GNS representation are denoted respectively by $\varphi_{x,t}$ and $\pi_{x,t}$ for $(x, t) \in \text{Per}(\sigma) \times \mathbb{T}$, and by φ_x and π_x for $x \in \text{Aper}(\sigma)$. For any subset Ω of X , the set of all pure state extensions of point evaluations $\mu_x \circ i_0^{-1}$ for $x \in \Omega$ will be denoted by $P(\Omega, A(\Sigma))$.

We will say that a representation of $A(\Sigma)$ is induced or comes from a point $x \in X$ if it is unitarily equivalent to the GNS representation corresponding to a pure state extension of a point evaluation at x .

For some dynamical systems $\Sigma = (X, \sigma)$ not all irreducible representations of $A(\Sigma)$ are induced from a point. A classical example is the dynamical system $\Sigma_\theta = (\mathbb{T}, \sigma_\theta)$ on the circle \mathbb{T} defined by the rotation σ_θ of \mathbb{T} by an angle $2\pi\theta$ with an irrational θ . The representation m of $C(\mathbb{T})$ by multiplication operators on $L^2(\mathbb{T}, \mu)$ with the Lebesgue measure μ , and the unitary operator $u_\theta : f(t) \mapsto f(t+\theta)$ corresponding to translation by θ , form a covariant representation giving rise to an irreducible representation of $A_\theta = A(\Sigma_\theta)$, which is not induced from a point for instance because in $L^2(\mathbb{T}, \mu)$, no such common eigensubspaces for $m(C(\mathbb{T}))$ as described in Theorem 2 can be found. This representation is infinite-dimensional.

In general, all non-zero irreducible representations of $A(\Sigma)$ which are not induced from a point are infinite-dimensional, since every finite-dimensional irreducible representations is induced from some periodic point. The later fact is well known in the theory of general transformation group C^* -algebras in much broader context. In the case of $A(\Sigma)$, this result has been proved in several ways in [KTW, Tom1, Tom2], and can be formulated as follows.

Proposition 3. *Every finite-dimensional representation is unitarily equivalent to the GNS representation $\pi_{x,\lambda}$ associated to a pure state extension of the point evaluation at some periodic point x . The dimension of the representation is equal to the period of x .*

Let π be a representation of $A(\Sigma)$ on a Hilbert space H_π . Then $\pi \circ i_0$ is a representation of $C(X)$ on the same Hilbert space, and $(\pi \circ i_0)(C(X)) = \pi(C(X)\delta^0)$, or in other words $(\pi \circ i_0)(C(X))$ consists of exactly the same operators as the image of the restriction of π onto the subalgebra $C(X)\delta^0$. The kernel J_π of $\pi \circ i_0$ is a closed two-sided ideal in $C(X)$, and hence there exists a closed subset X_π of X such that

$$J_\pi = k(X_\pi) = \{f \in C(X) \mid f(x) = 0 \text{ for all } x \in X_\pi\}.$$

Moreover, by the covariance relation, X_π is invariant under σ .

Let σ_π be the homomorphism of X_π obtained by restriction of σ onto X_π . The topological dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$ is called the dynamical system induced by π . If π is irreducible, then by Proposition 4.1.5 in [Tom1] the dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$ is topologically transitive in the sense that for any two nonempty open sets U and V in X_π

there exists some $n \in \mathbb{Z}$ such that $\sigma_\pi^n(U) \cap V \neq \emptyset$. The topological space X_π is compact, Hausdorff and satisfies the second axiom of countability as a topological subspace of the topological space X with these properties. So, transitivity of Σ_π is equivalent to existence of a point $x_0 \in X_\pi$ with the orbit $Orb_\sigma(x_0)$ dense in X_π .

The following Lemma, proved in [Tom3], is essential for description of irreducible representations of $A(\Sigma)$ and their relation to orbit structure of topological dynamical systems. We will give here elementary self-contained detailed proof, which shows explicitly how irreducible representations of $A(\Sigma)$ are described using the orbits of the dynamical system. The arguments in this proof go back to the work of Frobenius and Mackey on the induced representations of groups and systems of imprimitivity. The prove could be easily used to obtain a proof of Proposition 3 as well as the description of irreducible representations in Theorem 2 for type I algebras without mentioning relation to GNS representations associated to pure state extensions of point evaluations. Such description of irreducible representations, and the corresponding spectral decomposition theorems for arbitrary representations have been given in the works of Yu. S. Samoilenko, V. L. Ostrovskiy and E. Ye. Vaysleb [OS1, SaV, VSa] for type I algebras in terms of families of operators satisfying covariance commutation relations. There also representations by unbounded operators of more general crossed product type algebras, defined using not necessarily bijective mappings, have been considered together with examples of many concrete algebras studied in physics literature. The books by Yu. S. Samoilenko [Sa], and by V. L. Ostrovskiy and Yu. S. Samoilenko [OS2] contain many results and extensive bibliography in this direction. From the point of view of C^* -algebras however, the meaning of the representations associated to orbits becomes more clear when the relation to GNS representations and the corresponding pure states is described as in Theorem 2.

Lemma 4. *If the representation π is irreducible and infinite-dimensional, then X_π is infinite, or equivalently if X_π is finite then the irreducible representation π is finite-dimensional. Moreover, if X_π is finite, then the dimension of the representation space H_π is equal to the number of elements in X_π .*

Proof. Let π be an irreducible representation such that X_π is finite, and let us prove that in this case the irreducible representation π must be finite-dimensional.

If X_π is a finite subset of X then the topology on X_π induced from X is discrete, and hence $x_0 \in X_\pi$ is periodic and $X_\pi = \{x_0, \dots, \sigma^{p-1}(x_0)\}$, where p is the period of x_0 . As X is Hausdorff, there exists a function $\chi_0 \in C(X)$ such that

$$\chi_0(\sigma^j(x_0)) = \begin{cases} 1, & \text{if } j = 0 \\ 0, & \text{if } j \in \{1, \dots, p-1\}. \end{cases}$$

Since $\chi_0^2 - \chi_0$ and $\overline{\chi_0} - \chi_0$ belong to J_π , it follows that $\pi(\chi_0)^2 = \pi(\chi_0)$ and $\pi(\chi_0)^* = \pi(\chi_0)$, which means that $\pi(\chi_0)$ is a projection in $\pi(C(X))$. Moreover $\pi(\chi_0) \neq 0$ because $\chi_0 \notin J_\pi$. Since $\pi(\delta)$ is unitary, $\pi(\delta)^k \pi(\chi_0) (\pi(\delta)^*)^k$ is also a nonzero projection in $\pi(C(X))$ for all $k \in \{0, \dots, p-1\}$. By covariance relation $\pi(\delta)^k \pi(\chi_0) (\pi(\delta)^*)^k = \pi(\chi_0 \circ \sigma^{-k})$. Thus, as $(\chi_0 \circ \sigma^{-k})(\chi_0 \circ \sigma^{-j}) \in J_\pi$ when $k, j \in \{0, \dots, p-1\}$ and $k \neq j$, it follows that

$$\begin{aligned} \pi(\delta)^k \pi(\chi_0) (\pi(\delta)^*)^k \pi(\delta)^j \pi(\chi_0) (\pi(\delta)^*)^j &= \pi(\chi_0 \circ \sigma^{-k}) \pi(\chi_0 \circ \sigma^{-j}) = \\ &= \pi((\chi_0 \circ \sigma^{-k})(\chi_0 \circ \sigma^{-j})) = 0, \end{aligned}$$

which means that the projections $\pi(\delta)^k \pi(\chi_0) (\pi(\delta)^*)^k$ and $\pi(\delta)^j \pi(\chi_0) (\pi(\delta)^*)^j$ are mutually orthogonal if $k, j \in \{0, \dots, p-1\}$ and $k \neq j$. Therefore,

$$P_{\chi_0} = \sum_{j=0}^{p-1} \pi(\delta)^j \pi(\chi_0) (\pi(\delta)^*)^j = \sum_{j=0}^{p-1} \pi(\chi_0 \circ \sigma^{-j}) = \pi\left(\sum_{j=0}^{p-1} \chi_0 \circ \sigma^{-j}\right)$$

is a nonzero projection in $\pi(C(X))$ as well. Since $\sum_{j=0}^{p-1} \chi_0 \circ \sigma^{-j} \in C(X)$ and since $\pi(C(X))$ is commutative, $\pi(f)P_{\chi_0} = P_{\chi_0}\pi(f)$ for all $f \in C(X)$. Moreover, $\chi_0 \circ \sigma^{-p} - \chi_0 \in J_\pi$ because $\sigma^{-p}(x_0) = x_0$ and $\sigma^{-j}(x_0) \neq x_0$ for all $j \in \{1, \dots, p-1\}$. Thus $\pi(\delta)^p \pi(\chi_0) (\pi(\delta)^*)^p = \pi(\chi_0 \circ \sigma^{-p}) = \pi(\chi_0)$, and since $\pi(\delta)^* \pi(\delta) = I_{H_\pi}$ by unitarity of $\pi(\delta)$, it follows that

$$\begin{aligned} \pi(\delta)P_{\chi_0} &= \pi(\delta)\left(\sum_{k=0}^{p-1} \pi(\delta)^k \pi(\chi_0) (\pi(\delta)^*)^k\right) = \sum_{k=0}^{p-1} \pi(\delta)^{k+1} \pi(\chi_0) (\pi(\delta)^*)^{k+1} \pi(\delta) \\ &= \left(\sum_{k=0}^{p-1} \pi(\delta)^k \pi(\chi_0) (\pi(\delta)^*)^k\right) \pi(\delta) = P_{\chi_0} \pi(\delta). \end{aligned}$$

So, the projection P_{χ_0} commutes with $\pi(\delta)$ and $\pi(f)$ for all $f \in C(X)$, and therefore also with all elements of the C^* -algebra $\pi(\mathcal{A}(\Sigma))$ generated by $\pi(\delta)$ and $\pi(C(X)\delta)$. Since π is irreducible and P_{χ_0} is nonzero, $P_{\chi_0} = I_{H_\pi}$, or equivalently

$$H_\pi = P_{\chi_0} H_\pi = \bigoplus_{k=0}^{p-1} \pi(\delta)^k \pi(\chi_0) (\pi(\delta)^*)^k H_\pi.$$

Thus, H_π is finite-dimensional if and only if $\pi(\chi_0)H_\pi$ is finite-dimensional.

Let us show that irreducibility of π implies that $\pi(\chi_0)H_\pi$ is actually one-dimensional, and hence $\dim(H_\pi) = p$. Suppose to the contrary that $\dim(\pi(\chi_0)H_\pi) \neq 1$. The C^* -algebra Y_x generated by $\pi(C(X)) \cup \{\pi(\delta^{p^k}) \mid k \in \mathbb{Z}, k \geq 0\}$ is commutative and leaves $\pi(\chi_0)H_\pi$ invariant. Since all irreducible representations of commutative C^* -algebras are one-dimensional, by assumption $\dim(\pi(\chi_0)H_\pi) \neq 1$, the restriction of Y_x to $\pi(\chi_0)H_\pi$ is not irreducible on $\pi(\chi_0)H_\pi$, and thus there exists a proper closed subspace K of $\pi(\chi_0)H_\pi$ invariant with respect to the C^* -algebra Y_x .

Then $[\pi(\mathcal{A}(\Sigma))K] = [\{\pi(\delta^j)K \mid j \in \{0, \dots, p-1\}\}]$, where $[L]$ denotes the closed subspace of H_π spanned by $L \subseteq H_\pi$. Let P_K be the orthogonal projection onto K . Then the orthogonal complement of K in $\pi(\chi_0)H_\pi$ is $(\pi(\chi_0) - P_K)H_\pi = \pi(\chi_0)(I_{H_\pi} - P_K)H_\pi \subseteq \pi(\chi_0)H_\pi$. Moreover, the subspace $(\pi(\chi_0) - P_K)H_\pi$ is nonzero and orthogonal not only to K , but also to

$$\pi(\delta)^j K = \pi(\delta)^j \pi(\chi_0) K = \pi(\chi_0 \circ \sigma^{-j}) \pi(\delta)^j K \subseteq \pi(\chi_0 \circ \sigma^{-j}) H_\pi$$

for all $j \in \{1, \dots, p-1\}$, because the subspaces $\pi(\chi_0 \circ \sigma^{-j})H_\pi$ and $\pi(\chi_0)H_\pi$ are orthogonal for all $j \in \{1, \dots, p-1\}$. Therefore, $\pi(\chi_0 - P_K)H_\pi$ is orthogonal to the closed subspace $[\pi(\mathcal{A}(\Sigma))H_\pi]$ spanned by $\bigcup_{j=0}^{p-1} \pi(\delta)^j K$. Thus $[\pi(\mathcal{A}(\Sigma))H_\pi]$ is proper. But this contradicts to irreducibility of π . Hence, there is no proper closed subspace K in $\pi(\chi_0)H_\pi$ invariant with respect to Y_x . But this contradicts to existence of such subspace guaranteed

by the assumption $\dim(\pi(\chi_0)H_\pi) \neq 1$ and commutativity of Y_x . Thus $\dim(\pi(\chi_0)H_\pi) = 1$ and hence H_π is finite-dimensional and moreover $\dim(H_\pi) = p$. ■

The sets $Per(\sigma)$ and $Aper(\sigma)$ are equipped with topology from X , the set $Per(\sigma) \times \mathbb{T}$ is equipped with the product topology, and the spaces

$$\begin{aligned} P(Aper(\sigma), A(\Sigma)) &= \{\varphi_x = (\mu_x \circ i_0^{-1}) \circ E \mid x \in Aper(\sigma)\}, \\ P(Per(\sigma), A(\Sigma)) &= \{\varphi_{x,t} \mid (x,t) \in Per(\sigma) \times \mathbb{T}\} \end{aligned}$$

are equipped with the weak* topology from $P(A)$.

The mapping $T : Aper(\sigma) \bigcup (Per(\sigma) \times \mathbb{T}) \rightarrow P(X, A(\Sigma))$ sending each $x \in Aper(\sigma)$ to $\varphi_x = (\mu_x \circ i_0^{-1}) \circ E$ and each $(x,t) \in Per(\sigma) \times \mathbb{T}$ to $\varphi_{x,t}$ is a Borel isomorphism. The restrictions

$$\begin{aligned} T_{<\infty} : Per(\sigma) \times \mathbb{T} &\rightarrow \{\varphi_{x,t} \mid (x,t) \in Per(\sigma) \times \mathbb{T}\}, \\ T_\infty : Aper(\sigma) &\rightarrow \{\varphi_x = (\mu_x \circ i_0^{-1}) \circ E \mid x \in Aper(\sigma)\} \end{aligned}$$

of T are Borel isomorphisms as well.

Moreover, the following statements are true

Proposition 5. *The map*

$$T_n : Per_n(\sigma) \times \mathbb{T} \rightarrow \{\varphi_{x,t} \mid x \in Per_n(\sigma), t \in \mathbb{T}\}$$

obtained by restriction of the map T to $Per_n(\sigma) \times \mathbb{T}$ is a homeomorphism with respect to the weak topology in the pure state space.*

The map T_∞ is a homeomorphism into the pure state space of $A(\Sigma)$.

Proof. If a net $\{(y_\alpha, t_\alpha)\} \subseteq Per_n(\sigma) \times \mathbb{T}$ converges to a point $(y_0, t_0) \in Per_n(\sigma) \times \mathbb{T}$, then $\varphi_{y_\alpha, t_\alpha}(f) = f(y_\alpha)$ converges to $f(y_0)$ for every continuous function f . Moreover, $\varphi_{y_\alpha, t_\alpha}(\delta_\alpha^{nk}) = t_\alpha^k$ converges to $t_0^k = \varphi_{y_0, t_0}(\delta^{nk})$ for $k \in \mathbb{Z}$, and $\varphi_{y_\alpha, t_\alpha}(\delta^l) = \varphi_{y_0, t_0}(\delta^l) = 0$ if $l \in \mathbb{Z}$ is not divisible by n . Since, $\varphi_{y,t}(f\delta^m) = f(y)\varphi_{y,t}(\delta^m)$ for all $(y,t) \in Per_n(\sigma) \times \mathbb{T}$ and $m \in \mathbb{Z}$, and since the linear span of $\{f\delta^n \mid f \in C(X), n \in \mathbb{Z}\}$ is dense in $A(\Sigma)$, the net $\{\varphi_{y_\alpha, t_\alpha}$ converges to φ_{y_0, t_0} in the weak* topology. The converse continuity also follows easily from the above arguments. The assertion for T_∞ is obviously true because of the form of the extension $\varphi_x = (\mu_x \circ i_0^{-1}) \circ E$. ■

Let $\Phi : Aper(\sigma) \bigcup (Per(\sigma) \times \mathbb{T}) \rightarrow Irr(A(\Sigma))$, be the mapping defined as $\Phi(x) = \pi_x$ for $x \in Aper(\sigma)$ and $\Phi(x,t) = \pi_{x,t}$ for $(x,t) \in Per(\sigma) \times \mathbb{T}$. The restriction of Φ onto $Aper(\sigma)$ will be denoted by Φ_∞ , and the restrictions of Φ onto $Per(\sigma) \times \mathbb{T}$, $Per_n(\sigma) \times \mathbb{T}$ and $Per^n(\sigma) \times \mathbb{T}$ will be denoted respectively by $\Phi_{<\infty}$, Φ_n and $\Phi_{\leq n}$. Denote by X/\mathbb{Z} the orbit space of the dynamical system Σ . It follows from the statement 6 of Theorem 2 that the mapping

$$q \circ \Phi_\infty : Aper(\sigma) \rightarrow \widehat{\mathcal{A}(\Sigma)}_\infty = \widehat{\mathcal{A}(\Sigma)} \setminus \widehat{\mathcal{A}(\Sigma)}_{<\infty}$$

is constant on the sets $Orb_\sigma(x)$ for $x \in Aper(\sigma)$, and the mappings

$$q \circ \Phi_{<\infty} : Per(\sigma) \times \mathbb{T} \rightarrow \widehat{\mathcal{A}(\Sigma)}_{<\infty},$$

$$q \circ \Phi_n = q_n \circ \Phi_n : Per_n(\sigma) \times \mathbb{T} \rightarrow \widehat{\mathcal{A}(\Sigma)}_n,$$

$$q \circ \Phi_{\leq n} : Per^n(\sigma) \times \mathbb{T} \rightarrow \bigcup_{1 \leq k \leq n} \widehat{\mathcal{A}(\Sigma)}_k = \widehat{\mathcal{A}(\Sigma)}_{\leq n}$$

are constant on the sets $Orb_\sigma(x) \times t$ for (x, t) from $Per(\sigma) \times \mathbb{T}$, $Per_n(\sigma) \times \mathbb{T}$ and $Per^n(\sigma) \times \mathbb{T}$ respectively. So, there is the unique well-defined mapping

$$\Psi : (Aper(\sigma)/\mathbb{Z}) \bigcup ((Per(\sigma)/\mathbb{Z}) \times \mathbb{T}) \rightarrow \widehat{A(\Sigma)}$$

satisfying $\Psi \circ Q_\sigma = q \circ \Phi$ on $Aper(\sigma)$ and $\Psi \circ (Q_\sigma \times Id) = q \circ \Phi$ on $Per(\sigma) \times \mathbb{T}$. Since $Per_n(\sigma)$, $Per^n(\sigma)$ and their complements in $Per(\sigma)$ are invariant under the action of \mathbb{Z} for any positive integer n , there are well-defined mappings

$$\begin{aligned} \Psi_n &: (Per_n(\sigma)/\mathbb{Z}) \times \mathbb{T} \rightarrow \widehat{A(\Sigma)}_n \\ \Psi_{\leq n} &: (Per^n(\sigma)/\mathbb{Z}) \times \mathbb{T} \rightarrow \bigcup_{0 \leq k \leq n} \widehat{A(\Sigma)}_k \end{aligned}$$

obtained by restriction of Ψ respectively onto $(Per_n(\sigma)/\mathbb{Z}) \times \mathbb{T}$ and $(Per^n(\sigma)/\mathbb{Z}) \times \mathbb{T}$. The restrictions of Ψ onto $Aper(\sigma)/\mathbb{Z}$ and $(Per(\sigma)/\mathbb{Z}) \times \mathbb{T}$ will be denoted by Ψ_∞ and $\Psi_{<\infty}$.

For any C^* -algebra, the set $\hat{\mathcal{A}}$ is a topological space with the so called Jacobson or hull-kernel topology. For any dense subset D of \mathcal{A} , the sets

$$U_a = \{[\pi] \in \hat{\mathcal{A}} \mid \|\pi(a)\| > 1\}$$

with $a \in D$ form a base of the Jacobson topology in $\hat{\mathcal{A}}$. The set $\hat{\mathcal{A}}_\infty$ is a topological space with the topology obtained by restriction of the Jacobson topology from $\hat{\mathcal{A}}$.

The following proposition relates the periodic and aperiodic parts of orbit space to finite-dimensional and infinite-dimensional parts of representation space of the corresponding crossed product C^* -algebra. It has diverse applications both in topological dynamics and in the theory of C^* -algebras and their representations (see for example [AT], [KTW], [Tom1], [Tom2], [Tom3], [Tom4], [BrJ] and references there).

Proposition 6.

1. The space $\widehat{A(\Sigma)}_n$ of unitary equivalence classes of n -dimensional irreducible representations of $A(\Sigma)$ is homeomorphic to the product space $(Per_n/\mathbb{Z}) \times \mathbb{T}$.
2. The map Ψ_∞ is a homeomorphism from $Aper(\sigma)/\mathbb{Z}$ into the part of $\hat{\mathcal{A}}_\infty$ induced from aperiodic points.

Proof. The statement 1 of Proposition 6 has been proved in [KTW]. Another in a sense simpler proof, for the both statements 1 and 2, follows directly by combining Theorem 2, Proposition 5, Theorem 3.4.11 in [Dix2] stating that for any C^* -algebra \mathcal{A} the GNS mapping $\gamma : P(\mathcal{A}) \rightarrow \hat{\mathcal{A}}$ is continuous and open surjection onto $\hat{\mathcal{A}}$, the fact that the quotient mapping $Q_\sigma : X \rightarrow X/\mathbb{Z}$ is also a continuous open surjection, the observation that $\Psi \circ Q_\sigma = (q \circ \gamma) \circ T$ on $Aper(\sigma)$ and $\Psi \circ (Q_\sigma \times Id) = (q \circ \gamma) \circ T$ on $Per(\sigma) \times \mathbb{T}$ and on $Per_n(\sigma) \times \mathbb{T}$, and the fact presented as Theorem 9 in Chapter 3 in [Kel] stating that given a continuous surjection f from a topological space X onto a topological space Y with factor topology induced by f from the topology of X , a mapping g from Y onto a topological space Z is continuous if and only if the composition $g \circ f : X \rightarrow Z$ is continuous. ■

By Proposition 3, the mappings

$$\begin{aligned}\Psi_{<\infty} : (Per/\mathbb{Z}) \times \mathbb{T} &\rightarrow \bigcup_{1 \leq k < \infty} \widehat{\mathcal{A}(\Sigma)_k} = \widehat{\mathcal{A}(\Sigma)}_{<\infty} \\ \Psi_n : (Per_n(\sigma)/\mathbb{Z}) \times \mathbb{T} &\rightarrow \widehat{\mathcal{A}(\Sigma)}_n \\ \Psi_{\leq n} : (Per^n(\sigma)/\mathbb{Z}) \times \mathbb{T} &\rightarrow \widehat{\mathcal{A}(\Sigma)}_{\leq n}\end{aligned}$$

are surjections for any positive integer n .

There are topological dynamical systems $\Sigma = (x, \sigma)$ for which the mapping

$$\Psi_\infty : (Aper(\sigma)/\mathbb{Z}) \rightarrow \widehat{\mathcal{A}(\Sigma)}_\infty = \widehat{\mathcal{A}(\Sigma)} \setminus \widehat{\mathcal{A}(\Sigma)}_{<\infty}$$

is not surjective.

The following fact seems to be known among specialists, but we give here an elementary self-contained proof.

Proposition 7. *If X is compact Hausdorff topological space satisfying the second axiom of countability (i.e. compact metrizable space) and $\mathcal{A}(\Sigma)$ is a C^* -algebra of type I, then Ψ_∞ is surjective, which in other words means that every infinite-dimensional irreducible representation is unitarily equivalent to the GNS representation corresponding to the pure state extension of the point evaluation at some aperiodic point.*

Proof. By Lemma 4, if X_π is finite and π is irreducible, then H_π must be finite-dimensional with dimension $\dim(H_\pi)$ being equal to the number of elements in X_π . But we are considering the case when π is infinite-dimensional. Thus X_π is infinite, and since in this case $\sigma^k(x_0) \neq x_0$ for any nonzero integer k , the dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$ is effective, that is σ_π^k is not the identity mapping on X_π for any nonzero integer k . Since moreover, by irreducibility of π , the dynamical system Σ_π is topologically transitive, it is also topologically free in the sense that $Aper(\Sigma_\pi)$ is dense in X_π . By Proposition 5.2 in [Tom2], the subspace $Ker(\pi)$ coincides with the closure of the set

$$\left\{ \sum_{k=-n}^n f_k \delta^k \mid f_k \in C(X), n \in \mathbb{Z} \text{ and } f_k|_{X_\pi} = 0 \right\}.$$

For simplicity of notations π_{x_0} , H_{x_0} and ξ_{x_0} will be used to denote respectively the GNS representation π_{x_0} corresponding to the pure state extension of the point evaluation at x_0 , the Hilbert space and the GNS vector of the representation π_{x_0} .

Let us prove that $Ker(\pi)$ coincides with $Ker(\pi_{x_0})$. Indeed, as $(\pi_{x_0}(\delta)^*)^l$ is unitary and hence invertible for all $l \in \mathbb{Z}$, it follows that $\pi_{x_0}(\sum_{k=-n}^n f_k \delta^k) = 0$ if and only if $(\pi_{x_0}(\delta)^*)^l \pi_{x_0}(\sum_{k=-n}^n f_k \delta^k) = 0$ for all $l \in \mathbb{Z}$ satisfying $-n \leq l \leq n$. The Hilbert space H_{x_0} coincides with a closed linear span of $\{\pi_{x_0}^j(\delta) \xi_{x_0} \mid j \in \mathbb{Z}\}$. Thus the last condition is equivalent to

$$((\pi_{x_0}(\delta)^*)^l \pi_{x_0}(\sum_{k=-n}^n f_k \delta^k) \pi_{x_0}(\delta)^j \xi_{x_0}, \pi_{x_0}(\delta)^j \xi_{x_0}) = 0$$

being satisfied for all $j \in \mathbb{Z}$ and all $l \in \mathbb{Z}$ such that $-n \leq l \leq n$. Since

$$\begin{aligned} & ((\pi_{x_0}(\delta)^*)^l \pi_{x_0}(\sum_{k=-n}^n f_k \delta^k) \pi_{x_0}(\delta)^j \xi_{x_0}, \pi_{x_0}(\delta)^j \xi_{x_0}) = \\ &= \sum_{k=-n}^n ((\pi_{x_0}(\delta)^*)^l \pi_{x_0}(f_k) \pi_{x_0}(\delta)^{k+j} \xi_{x_0}, \pi_{x_0}(\delta)^j \xi_{x_0}) = \\ &= \sum_{k=-n}^n f_k (\sigma^{k+j}(x_0)) (\pi_{x_0}(\delta)^{-l+k+j} \xi_{x_0}, \pi_{x_0}(\delta)^j \xi_{x_0}) = f_l (\sigma^{l+j}(x_0)) \end{aligned}$$

for all $j \in \mathbb{Z}$ and all $l \in \mathbb{Z}$ such that $-n \leq l \leq n$, the element $\sum_{k=-n}^n f_k \delta^k$ belongs to $\text{Ker}(\pi_{x_0})$ if and only if $f_l(\sigma^k(x_0)) = 0$ for all $k \in \mathbb{Z}$ and all $l \in \mathbb{Z}$ such that $-n \leq l \leq n$, or equivalently if and only if $f_l|_{\text{Orb}_\sigma(x_0)} = 0$ for all l such that $-n \leq l \leq n$. This, in its turn, is equivalent to $f_l|_{X_\pi} = 0$, because $\text{Orb}_\sigma(x_0)$ is dense in X_π and $f_l \in C(X)$ for all l such that $-n \leq l \leq n$.

Since $\{\sum_{k=-n}^n f_k \delta^k \mid f_k \in C(X), n \in \mathbb{Z}\}$ is dense in $A(\Sigma)$, the set

$$\begin{aligned} & \text{Ker}(\pi_{x_0}) \cap \{\sum_{k=-n}^n f_k \delta^k \mid n \in \mathbb{Z}, f_k \in C(X)\} = \\ &= \{\sum_{k=-n}^n f_k \delta^k \mid n \in \mathbb{Z}, f_k \in C(X) \text{ and } f_k|_{X_\pi} = 0\} \end{aligned}$$

is dense in $\text{Ker}(\pi_{x_0})$, and since it is also dense in $\text{Ker}(\pi)$, it follows that $\text{Ker}(\pi) = \text{Ker}(\pi_{x_0})$. If $\mathcal{A}(\Sigma)$ is of type I, then by Theorem 4.3.7 in [Dix2] mentioned before the equality $\text{Ker}(\pi) = \text{Ker}(\pi_{x_0})$ implies that π and π_{x_0} are unitarily equivalent. Since π is an arbitrary irreducible representation and x_0 is an aperiodic point, the mapping Ψ_∞ is surjective as has been claimed. ■

Proposition 7 and Proposition 3 combined yield the following result.

Proposition 8. *If X is a compact metrizable space and the C^* -algebra $A(\Sigma)$ is of type I, then the mapping Ψ is surjective, or in other words any irreducible representation is unitarily equivalent to the GNS representation of a pure state extension of a point evaluation.*

For convenience we will say simply that an irreducible representation of $A(\Sigma)$ comes from a point if it is unitarily equivalent to the GNS representation corresponding to the pure state extension of the point evaluation at this point.

3 Dynamical systems of type I

We are now ready to proceed with the proof of the Theorem 1. After the proof we will present also some consequences of the theorem and questions for further investigation.

Proof. (Theorem 1) In [AT] it was proved that for compact Hausdorff topological space X satisfying the second axiom of countability the C^* -algebra $A(\Sigma)$ is of type I if and only if $\mathcal{C}(\sigma) = \text{Per}(X)$, that is if and only if all recurrent points of σ are periodic.

By propositions 6 and 7, when $\mathcal{A}(\Sigma)$ is a C^* -algebra of type I, the mapping

$$\Psi_\infty : \text{Aper}(\sigma)/\mathbb{Z} \rightarrow \widehat{\mathcal{A}(\Sigma)_\infty}$$

is a homeomorphism. If \mathcal{A} is a C^* -algebra of type I, then $\hat{\mathcal{A}}$ is a T_0 -space, that is for any pair of distinct points in $\hat{\mathcal{A}}$ there is an open set in $\hat{\mathcal{A}}$ containing only one of these two points. This fact is contained in Propositions 3.1.3, 3.1.6 and Theorem 4.3.7 in [Dix2]. So if $\mathcal{A}(\Sigma)$ is of type I, then $\widehat{\mathcal{A}(\Sigma)}$ is a T_0 -space. Hence $\widehat{\mathcal{A}(\Sigma)}_\infty$ is a T_0 -space as a subspace of T_0 -space. Thus $Aper(\Sigma)/\mathbb{Z}$ is also a T_0 -space, since it is homeomorphic to $\widehat{\mathcal{A}(\Sigma)}_\infty$.

By Proposition 3.6.3 in [Dix2], for any C^* -algebra \mathcal{A} the set $\hat{\mathcal{A}}_{\leq n}$ is closed in $\hat{\mathcal{A}}$ for any positive integer n . So, $\hat{\mathcal{A}} \setminus \hat{\mathcal{A}}_{\leq n}$ is open for any positive integer n , and since

$$\hat{\mathcal{A}}_\infty = \hat{\mathcal{A}} \setminus \bigcup_{n>0} \hat{\mathcal{A}}_n = \bigcap_{n>0} (\hat{\mathcal{A}} \setminus \hat{\mathcal{A}}_n),$$

the set $\hat{\mathcal{A}}_\infty$ is a G_δ -set in $\hat{\mathcal{A}}$. In particular, $\widehat{\mathcal{A}(\Sigma)}_\infty$ is a G_δ -set in $\widehat{\mathcal{A}(\Sigma)}$.

If $\mathcal{A}(\Sigma)$ is of type I, then the GNS mapping $\gamma : P(X, \mathcal{A}(\Sigma)) \rightarrow \widehat{\mathcal{A}(\Sigma)}$ is continuous surjection. Hence, $\gamma^{-1}(\widehat{\mathcal{A}(\Sigma)}_\infty)$ is a G_δ -set in $P(X, \mathcal{A}(\Sigma))$, and since

$$T : Aper(\sigma) \bigcup (Per(\sigma) \times \mathbb{T}) \rightarrow P(X, \mathcal{A}(\Sigma))$$

is a homeomorphism, $Aper(\sigma) = (T^{-1} \circ \gamma^{-1})(\widehat{\mathcal{A}(\Sigma)}_\infty)$ is a G_δ -set in $Aper(\sigma) \bigcup (Per(\sigma) \times \mathbb{T})$ and hence a G_δ -set in the polish space $X = Aper(\sigma) \bigcup Per(\sigma)$. The conclusion that $Aper(\sigma)$ is a G_δ -set in X can actually be obtained in much easier way, since it follows simply by definition of G_δ -set from the observation that

$$Aper(\sigma) = \bigcap_{n>0} (X \setminus Per^n(\sigma))$$

and from the fact that $Per^n(\sigma)$ is a closed subset of X for any positive integer n , because it is a union of finite number of closed subsets $Per_k(\sigma)$ for $1 \leq k \leq n$. A G_δ -set in a polish space is itself polish. Since $Aper(\sigma)$ is polish and $Aper(\sigma)/\mathbb{Z}$ is a T_0 -space, by Theorem 2.6 from [Ef1] the space $Aper(\sigma)/\mathbb{Z}$ is countably separated.

Let us consider $Per(\sigma)/\mathbb{Z}$. If \mathcal{A} is a separable C^* -algebra of type I, then $\hat{\mathcal{A}}$ is countably separated. In particular, $\widehat{\mathcal{A}(\Sigma)}_{<\infty}$ is countably separated, and hence $\hat{\mathcal{A}(\Sigma)}_n$ is countably separated for any positive integer n .

By Theorem A in [KTW], $\hat{\mathcal{A}(\Sigma)}_n$ is homeomorphic to $(Per_n(\sigma)/\mathbb{Z}) \times \mathbb{T}$. Thus, $(Per_n(\sigma)/\mathbb{Z}) \times \mathbb{T}$ is countably separated for any positive integer n . Hence, $Per_n(\sigma)/\mathbb{Z}$ is countably separated for any positive integer n . Since $Per(\sigma)/\mathbb{Z} = \bigcup_{n>0} (Per_n(\sigma)/\mathbb{Z})$ with

the sets in the union being disjoint, $Per(\sigma)/\mathbb{Z}$ is countably separated. As $Aper(\sigma)/\mathbb{Z}$ and $Per(\sigma)/\mathbb{Z}$ are countably separated, the disjoint union $X/\mathbb{Z} = (Aper/\mathbb{Z}) \bigcup (Per(\sigma)/\mathbb{Z})$ is countably separated too. By Theorem 2.9 from [Ef1] there exists a Borel measurable section for $\sigma = (X, \sigma)$, that is a Borel subset M of X which has exactly one point in common with the orbit $Orb_\sigma(x)$ of every point $x \in X$.

By corollary 2.1 from [AT], the C^* -algebra $\mathcal{A}(\Sigma)$ is of type I if and only if $\mathcal{C}(\sigma) = Per(\sigma)$. So, if $\mathcal{C}(\sigma) = Per(\sigma)$, then $\mathcal{A}(\Sigma)$ is of type I and hence there exists a Borel measurable section for $\Sigma = (X, \sigma)$.

Let us now prove that if there exists a Borel measurable section for a dynamical system $\Sigma = (X, \sigma)$, then $\mathcal{C}(\sigma) = \text{Per}(\sigma)$.

Suppose that there exists $\Sigma = (X, \sigma)$ such that there is a Borel measurable section, but $\mathcal{C}(\sigma) \neq \text{Per}(\sigma)$. The last condition means that $\mathcal{C}(\sigma) \setminus \text{Per}(\sigma)$ is not empty. When X is a compact metric space and $x \in \mathcal{C}(\sigma) \setminus \text{Per}(\sigma)$, there exists a non-atomic quasi-invariant ergodic measure μ_x supported on the closure $\overline{\text{Orb}_\sigma(x)}$ of the orbit of x . The prove of this known fact could be for example recovered from general arguments in the paper [Efl] dealing with general transformation group C^* -algebras (see also [Nadk]). A detailed constructive proof of this fact in our case of the action by a single homeomorphism on a compact metric space is given in [Tom4].

Let M be a Borel measurable section for Σ . This means that M is a Borel measurable subset of X intersecting every orbit of Σ at a single point. Then

$$X = \bigcup_{n \in \mathbb{Z}} \sigma^n(M) \quad (\text{disjoint union}).$$

Thus, as μ_x is quasi-invariant, M must have a positive measure, and since μ_x is non-atomic, the set M can be split into the disjoint union of Borel sets M_1 and M_2 of positive measures. Then the union $\bigcup_{n \in \mathbb{Z}} \sigma^n(M_1)$ is a non-trivial invariant subset for μ . But this contradicts to ergodicity of μ_x . Thus $\mathcal{C}(\sigma) = \text{Per}(\sigma)$. ■

The importance of quasi-invariant measures on closures of orbits can be seen in the following constructive proof of the previously mentioned fact that if $\mathcal{C}(\sigma) \neq \text{Per}(\sigma)$, then $\mathcal{A}(\Sigma)$ is not of type I. Indeed, by Proposition 8, it is enough to point out the representation of $\mathcal{A}(\Sigma)$ not induced from a point of X . Take $y \in \mathcal{C}(\sigma) \setminus \text{Per}(\sigma)$, and let μ_y be a non-atomic quasi-invariant ergodic measure supported on the closure $\overline{\text{Orb}_\sigma(y)}$ of the orbit of y . Let $\mu_{y,\sigma} = \mu_y \circ \sigma^{-1}$ denote the measure defined as $\mu_{y,\sigma}(E) = \mu_y(\sigma^{-1}(E))$. Let π_0 be the representation of $C(X)$ on a Hilbert space $H = L^2(X, \mu_y)$ as multiplication operators. Define the unitary operator u as

$$(uf)(x) = \left(\frac{d\mu_{y,\sigma}}{d\mu_y} \right)^{\frac{1}{2}}(x) f(\sigma^{-1}(x)).$$

Then $n \mapsto u^n$ defines a unitary representation of the group \mathbb{Z} . Since

$$(u^*f)(x) = \left(\frac{d(\mu_y \circ \sigma)}{d\mu_y} \right)^{\frac{1}{2}}(x) f(\sigma(x)),$$

the pair $\{\pi_0, u\}$ becomes a covariant representation of $\{C(X), \alpha, \mathbb{Z}\}$, and as μ_y is ergodic measure, the representation π of $\mathcal{A}(\Sigma)$ generated by $\{\pi_0, u\}$ is irreducible. Suppose that the representation π is unitarily equivalent to an irreducible representation induced by a point $x_0 \in X$. Then according to Theorem 2 there would exist a non-zero subspace H_0 such that $\pi(f)\xi = f(x_0)\xi$ for all $\xi \in H_0$ and all $f \in C(X)$. It follows that for a non-zero function $g \in H_0$ we have

$$(f(x) - f(x_0))g(x) = 0 \quad \text{a.e. with respect to } \mu_y$$

for every $f \in C(X)$. Hence there is a measurable set E of positive measure $\mu_y(E) > 0$ on which any continuous function $f \in C(X)$ is the constant $f(x_0)$. But this contradicts to

the property of μ_y being non-atomic. Thus the representation π is not induced from the points of X .

Combining all previous results and considerations and choosing some of the equivalent conditions from [Ef1], [Dix2], [Dix2], [G11] and [AT], we obtain the following theorem containing as one of the statements Theorem 1 when X is compact metrizable space.

Theorem 9. *Let X be a compact metrizable space and σ be a homeomorphism of X , then the following assertions are equivalent:*

1. $\mathcal{A}(\Sigma)$ is a C^* -algebra of type I.
2. Any irreducible representation of $\mathcal{A}(\Sigma)$ is unitarily equivalent to the GNS representation corresponding to the pure state extension of a point evaluation at some point of X .
3. Two irreducible representations of $\mathcal{A}(\Sigma)$ are unitarily equivalent if and only if they have the same kernel.
4. The sets $\mathcal{C}(\sigma)$ and $\text{Per}(\sigma)$ coincide, that is all recurrent points are periodic points.
5. The orbit space of Σ is a T_0 -space.
6. There exists a Borel measurable section for Σ , i.e. a Borel measurable set intersecting every orbit of Σ at a single point.

Properties 1, 2 and 3 in Theorem 9 are concerned with the algebra $\mathcal{A}(\Sigma)$ and its representations, whereas properties 4, 5 and 6 are the most relevant for dynamical systems. The property 5 provides some information on the orbit space, but is of limited use since checking whether the orbit space of a dynamical system is T_0 -space often is a difficult task requiring deep understanding of the structure of the orbit space. The equivalence of properties 4 and 6 is the statement of Theorem 1. These two conditions are the most useful ones from the point of view of dynamical systems, since for many dynamical systems either the sets of all periodic points and of recurrent points can be computed or at least compared, or a Borel measurable section can be explicitly described or proved to be non-existent. For actions of general groups and groupoids, a number of other conditions equivalent to those in Theorem 9 have been studied in [Ef1], [Ef2], [Ef3], [G11], [G12], [MRW], [Ram] containing also historical comments and other related references.

The Theorem 1 suggests the following definition.

Definition 1. Let $\Sigma = (X, \sigma)$ be a topological dynamical system consisting of a topological space and of a homeomorphism σ of X . Then Σ will be called a dynamical system of *dynamical type I* if $\mathcal{C}(\sigma) = \text{Per}(\sigma)$. If the transformation group C^* -algebra $\mathcal{A}(\Sigma)$ is of type I, then (X, σ) will be called a dynamical system of *C^* -algebraic type I*. Finally, if there exists a Borel measurable section for (X, σ) , then (X, σ) will be called a dynamical system of *Borel type I*.

Remark 6. By the previously discussed results, if the topological space X is a compact metrizable space, that is a compact Hausdorff space satisfying the second axiom of countability, then for a dynamical system $\Sigma = (X, \sigma)$ generated by a homeomorphism

σ , all three properties of being dynamical type I, Borel type I or C^* -algebraic type I are equivalent.

The circle is a compact metric space with respect to the arc length distance. The following statement is true with regard to homeomorphisms of the circle.

Proposition 10. *Let σ be a homeomorphism of the circle.*

1. *If σ is orientation preserving, then it is of dynamical, C^* -algebraic or Borel type I if and only if it admits periodic points.*
2. *An orientation reversing homeomorphism is always of type I.*

The first assertion can be deduced from the fact that periods of all periodic points are the same and from consideration of connected components of aperiodic points if such points exist. The second assertion follows from the first assertion. Indeed, any orientation reversing homeomorphism of the circle has a fixed point. This fixed point is also a fixed point for σ^2 . Therefore, since σ^2 is orientation preserving, it is of dynamical, C^* -algebraic and Borel type I by the first assertion of the proposition. Thus the set of periodic points for σ^2 coincides with the set of its recurrent points. But σ and σ^2 have the same set of periodic points and the same set of recurrent points, and hence σ is also of dynamical, C^* -algebraic and Borel type I.

In general, if σ is a homeomorphism of a compact Hausdorff space, then the homeomorphisms σ and σ^m have the same set of periodic points and the same set of recurrent points for any positive integer m . The equality $\text{Per}(\sigma) = \text{Per}(\sigma^m)$ and the inclusion $\mathcal{C}(\sigma^m) \subseteq \mathcal{C}(\sigma)$ follows directly from the definition of periodic points and of recurrent points respectively. The opposite inclusion $\mathcal{C}(\sigma) \subseteq \mathcal{C}(\sigma^m)$ is a result proved by W. H. Gottschalk [Got] and in more general form by P. Erdős and A. H. Stone [EST] (see also [Vr], Ch. II, 10.8, page 142). Consequently, if σ is a homeomorphism of a compact Hausdorff space, then the dynamical system (X, σ) is of dynamical type I if and only if the dynamical system (X, σ^m) is of dynamical type I for some and thus for all positive integers m . If moreover X satisfies the second axiom of countability, then the "dynamical type I" can be replaced in the last statement by "Borel type I", as well as by " C^* -algebraic type I". One half of this statement about " C^* -algebraic type I" naturally corresponds to the standard fact, that every C^* -subalgebra of C^* -algebra of type I is also of type I. The another half of the statement means, that under the stated conditions on the topological space X , the crossed product C^* -algebra $\mathcal{A}(X, \sigma)$, which is associated with the dynamical system (X, σ) , is of type I if and only if the crossed product C^* -algebra $\mathcal{A}(X, \sigma^m)$ associated with the dynamical system (X, σ^m) , is of type I for some and thus for any positive integer m . Note that the C^* -algebra $\mathcal{A}(X, \sigma^m)$ can be embedded into $\mathcal{A}(X, \sigma)$ as the C^* -subalgebra generated by $C(X)\delta^0$ and $\{\delta^{mk} | k \in \mathbb{Z}\}$.

In the new terminology Theorem 1 states that a topological dynamical system (X, σ) with locally compact Hausdorff topological space X whose Alexandroff's one-point compactification satisfies the second axiom of countability is of dynamical type I if and only if it is of Borel type I. We do not know to which level the condition of X being Hausdorff can be weakened. The situation is the same with the condition of X being locally compact.

It could be also of interest to describe exactly for which compactifications other than the Alexandroff's one-point compactification the conclusion of Theorem 1 remains true.

In the proof of Theorem 1 we used the result from [AT] stating, in the new terminology, that a dynamical system (X, σ) with metrizable compact topological space X is of dynamical type I if and only if it is of C^* -algebraic type I. It is not known to us whether conditions of metrizability and compactness in this statement can be removed or to what extent these conditions can be relaxed. When X is a compact Hausdorff space and the requirement of X satisfying the second axiom of countability (i.e. metrizability) is dropped, it has been showed in [Tom4] that if the C^* -algebra $\mathcal{A}(\Sigma)$ is of type I, then $\mathcal{C}(\sigma) = \text{Per}(\sigma)$, or using our terminology if Σ is of C^* -algebraic type I, then it is of dynamical type I. Whether the converse is true in such a generality is not clear. It is also unclear whether the second axiom of countability is both sufficient and necessary, or if not what is the sufficient and necessary condition for equivalence of C^* -algebraic type I and dynamical type I. If X is compact, then $\mathcal{A}(\Sigma)$ is a unital C^* -algebra and thus it contains the largest ideal $K(\sigma)$ of type I for which the quotient algebra $\mathcal{A}(\Sigma)/K(\sigma)$ has no type I portion. In [Tom4], it was shown that firstly, if X is compact Hausdorff space, then

$$K(\sigma) \subseteq \bigcap_{x \in \mathcal{C}(\sigma) \setminus \text{Per}(\sigma)} \text{Ker}(\pi_x), \quad (11)$$

where π_x as before denotes the irreducible GNS representation corresponding to the pure state extension of the point evaluation at aperiodic point x , and secondly if the set $\mathcal{C}(\sigma) \setminus \text{Per}(\sigma)$ is dense in X , then $\mathcal{A}(\Sigma)$ becomes antiliminal C^* -algebra, which means that the largest ideal $K(\sigma)$ of type I is trivial. In [Tom4] it was shown that if X is metrizable, then the equality in (11) holds, and the density of $\mathcal{C}(\sigma) \setminus \text{Per}(\sigma)$ in X becomes equivalent to $\mathcal{A}(\Sigma)$ being antiliminal. What condition on X is necessary and sufficient for this equivalence or for the equality in (11) is unknown. These facts however suggest that some of the mentioned problems could possibly be resolved in terms of more refined relation between the ideal $K(\sigma)$ and the kernels of irreducible representations of $\mathcal{A}(\Sigma)$.

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